On Some Covering Properties of B-open sets

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1. Abstract

In this paper we introduce and study the concepts of b-open set, b-continuous functions, then we also study the concepts of b-compact subsets and study some new characterizations of b-separation axioms such as \( b-T_2 \). Then we discuss the relations between the b-continuous functions and these concepts.

Keywords

b-open set, b-compact, b-open cover, b-closed sets, b-continuous

2. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Levine [7] introduced the notion of semi-open sets and semi-continuity in topological spaces. Andrijevic [2] introduced a class of generalized open sets in topological spaces. Mashhour [9] introduced pre open sets in topological spaces. The class of b-open sets is contained in the class of semi-open and pre-open sets. In this paper we discuss the covering properties of b-sets and b-continuous functions. All through this paper \( (X, \tau) \) and \( (Y, \sigma) \) stand for topological spaces with no separation assumed, unless otherwise stated. The closure of A and the interior of A will be denoted by Cl(A) and Int(A), respectively.

3. Preliminaries

**Definition 3.1** A subset A of a space X is said to be [2],[10]:

1. Semi-open if \( A \subseteq \text{Cl}(\text{Int}(A)) \)
2. Pre open if \( A \subseteq \text{Int}(\text{Cl}(A)) \)
3. \( \alpha \)-open if \( A \subseteq \text{Int}((\text{Cl}(\text{Int}(A))) \)
4. \( \beta \)-open if \( A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A))) \)
5. b-open if \( A \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Int}(\text{Cl}(A)) \)
**Definition 3.2.** A function \( f : X \rightarrow Y \) is called 

1. semi continuous if \( f^{-1}(V) \) is semi open in \( X \) for each open set \( V \) of \( Y \).

2. pre continuous if \( f^{-1}(V) \) is pre open in \( X \) for each open set \( V \) of \( Y \).

3. \( \alpha \)-continuous if \( f^{-1}(V) \) is \( \alpha \) -open in \( X \) for each open set \( V \) of \( Y \).

4. \( \beta \)-continuous if \( f^{-1}(V) \) is \( \beta \) -open in \( X \) for each open set \( V \) of \( Y \).

5. \( b \)-continuous if \( f^{-1}(V) \) is \( b \)-open in \( X \) for each open set \( V \) of \( Y \).

**Definition 3.3** \([10]\) A space \( X \) is a \( b \)-T\(_2\) space iff for each \( x, y \in X \) such that \( x \neq y \) there are \( b \)-open sets \( U, V \subset X \) so that \( x \in U, y \in V \) and \( U \cap V = \emptyset \).

**4. Covering Properties**

**Definition 4.1**

Let \( \{G_\alpha : \alpha \in \Delta\} \) be a family of \( b \)-open sets of the space \( X \). the family \( \{G_\alpha : \alpha \in \Delta\} \) covers \( X \) if \( X \subseteq \bigcup_{\alpha \in \Delta} G_\alpha \).

**Definition 4.2**

A space \( X \) is called a \( b \)-compact space if each \( b \)-open cover of \( X \) has a finite subcover for \( X \).

**Theorem 4.3**

Let \( A \) be a \( b \)-compact subset of the \( b \)-T\(_2\) space \( X \) and \( \notin A \). then there exist two disjoint \( b \)-open sets \( U \) and \( V \) containing \( x \) and \( A \), respectively.

**Proof :**

Let \( y \in A \), since \( X \) is \( b \)-T\(_2\) space there exist two \( b \)-open sets \( U_x, V_y \in X \) such that \( x \in U_x, y \in V_y, U_x \cap V_y = \emptyset \), the family \( \bigcup \{A \cap V_y : y \in A\} \) is open cover of \( A \) has a finite subcover \( \{A \cap V_y_1, A \cap V_y_2, \ldots, A \cap V_y_n\} \), thus \( U = U_y_1 \cup U_y_2 \cup \ldots \cup U_y_n \).
**Theorem 4.4**

If \( X \) is b-T\(_2\) space and \( A \) is a b-open subset, if \( A \) is b-compact then \( A \) is a b-closed.

**Proof:**

Let \( x \in X - A \), by the theorem 4.3 there exist two b-open sets \( U \) and \( V \) such that \( x \in U, A \subseteq V, U \cap V = \emptyset \), thus \( x \in U \subseteq X - V \subseteq X - A \), which implies \( X - A \) is b-open so that \( A \) is b-closed.

**Theorem 4.5**

Let \( A \) and \( B \) be a two b-compact subsets of the b-T\(_2\) space \( X \), then there exist disjoint b-open sets \( U \) and \( V \) containing \( A \) and \( B \), receptively.

**Proof:**

Let \( b \in B \), since \( A \) is a b-compact subset and b-open in \( X \), there exist two b-open sets \( U_b, V_b \) such that \( U_b \cap V_b = \emptyset ; b \in V_b, A \subseteq U_b \), so \( \beta = \{ B \cap V_b; b \in B \} \) is a b-open cover of \( B \), since \( B \) is b-compact subset there exist finite subcover \( \{ B \cap V_m ; 1 \leq i \leq n \} \) from \( \beta \).

Let \( U = \bigcap_{i=1}^{n} U_{b_i}, V = \bigcup_{i=1}^{n} V_{b_i}, \) thus \( A \subseteq U, B \subseteq V, U \cap V = \emptyset \).

**Theorem 4.5**

Let \( f : (X, \tau) \rightarrow (Y, \rho) \) be a continuous surjection open function, if \( X \) is a b-compact then \( Y \) is a b-compact.

**Proof:**

Let \( \beta = \{ V_{\alpha} : \alpha \in \Delta \} \) be a b-open cover of \( Y \), then \( L = \{ f^{-1}(V_{\alpha}) : \alpha \in \Delta \} \) is a b-open cover of \( X \).since \( X \) is a b-compact space, there exist a finite subcover from \( L \) to the space \( X \) such that

\[
X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}), \text{ thus } Y = f(X) \subseteq f\left( \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}) \right) = f\left( f^{-1}\left( \bigcup_{i=1}^{n} (V_{\alpha_i}) \right) \right) = \bigcup_{i=1}^{n} (V_{\alpha_i})
\]

Hence \( Y \subseteq \bigcup_{i=1}^{n} (V_{\alpha_i}) \), this shows \( Y \) is a b-compact.
Corollary 4.6

B-compactness is a topological property

Proof:

The proof from theorem Theorem 4.5.

Definition 4.7:

A family of sets $\beta$ has “finite intersection property” if every finite subfamily of $\beta$ has a nonempty intersection.

Theorem 4.5

A topological space is compact if and only if any collection of its closed sets having the finite intersection property has non-empty intersection.

Proof:

Suppose X is b-compact, i.e., any collection of b-open subsets that cover X has a finite collection that also cover X. Further, suppose $\{G_\alpha : \alpha \in \Delta\}$ is an arbitrary collection of b-closed subsets with the finite intersection property. We claim that $\bigcap_{\alpha \in \Delta} G_\alpha \neq \emptyset$ is non-empty. Suppose otherwise, i.e., suppose $\bigcap_{\alpha \in \Delta} G_\alpha = \emptyset$. Then

$$\bigcup_{\alpha \in \Delta} (X - G_\alpha) = X - \left( \bigcap_{\alpha \in \Delta} G_\alpha \right) = X - \emptyset = X.$$ Since each $G_\alpha$ is b-closed, the collection $\{X - G_\alpha : \alpha \in \Delta\}$ is an b-open cover for X. By compactness, there is a finite subcover L such that

$$X = \bigcup_{i=1}^n (X - G_{\alpha_i}).$$ But then $\bigcap_{i=1}^n G_{\alpha_i} = \bigcap_{i=1}^n (X - (X - G_{\alpha_i})) = X - \left( \bigcup_{i=1}^n (X - G_{\alpha_i}) \right) = X - X = \emptyset,$ which contradicts the finite intersection property of $\{G_\alpha : \alpha \in \Delta\}$.

Conversely, take the hypothesis that every family of a b-closed sets in X having the finite intersection property has a nonempty intersection. we are to show X is compact. Let $\{G_\alpha : \alpha \in \Delta\}$ be any b-open cover of X. then $\{X - G_\alpha : \alpha \in \Delta\}$ is a family of b-closed sets such that $\bigcap X - G_\alpha = X - \left( \bigcup_{\alpha \in \Delta} G_\alpha \right) = X - X = \emptyset.$ Consequently, our hypothesis implies the family

$\{X - G_\alpha : \alpha \in \Delta\}$ does not have the finite intersection property. Therefore, there is some finite sub collection $\{X - G_{\alpha_i} : i = 1, 2, 3, \ldots, n\}$ such that $\bigcap_{i=1}^n X - G_{\alpha_i} = \emptyset$ and hence

$$X = \bigcup_{i=1}^n G_{\alpha_i} = \bigcup_{i=1}^n (X - (X - G_{\alpha_i})) = X - \left( \bigcap_{i=1}^n (X - G_{\alpha_i}) \right) = X - \emptyset = X.$$ Thus $X = \bigcup_{i=1}^n G_{\alpha_i}$, implying X is b-compact.
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References


