On A New Sequence Space of Non-Absolute Type and Inclusion Relations

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Abstract: In this paper, we introduce the space $\ell^\infty(\Delta^\lambda_u)$, which is BK-spaces of non-absolute type and prove that $\ell^\infty(\Delta^\lambda_u)$ is linearly isomorphic to the $\ell^\infty$, where $\lambda=(\lambda_k)_{k=0}^\infty$ is a strictly increasing sequence of positive reals tending to $\infty$. We also give some inclusion relations.

Keywords: Sequence spaces of non-absolute type, BK-space, Difference Sequence Spaces.

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1. Introduction
Let $w$ denote the spaces of all sequences (real or complex). A sequence space is defined to be linear space of real or complex number.

Let $X$ be a sequence space. If $X$ is a Banach space and

$$\tau_k : X \rightarrow C, \quad \tau_k (x) = x_k \quad (k = 1, 2, 3...).$$

is a continuous for all $k$, $X$ is called a BK-space.

We shall write $\ell^\infty$, $c$ and $c_0$ for the sequence spaces of all bounded, convergent and null sequences, respectively, which are BK-spaces with the same norm given by

$$\|x\|_\infty = \sup_k |x_k|$$

for all $k \in \mathbb{N}$.

The notion of difference sequence spaces was introduced by Kızmaz [1] as follows:

$$\ell^\infty(\Delta) = \{ x \in w : \Delta x \in \ell^\infty \},$$

$$c(\Delta) = \{ x \in w : \Delta x \in c \},$$

$$c_0(\Delta) = \{ x \in w : \Delta x \in c_0 \}.$$

Further more, he proved these spaces are BK-spaces with norm given by

$$\|x\|_\infty = \|x\| + \|\Delta x\|_\infty.$$

M. Mursaleen and A.K. Noman [5] introduced the sequence spaces $\ell^\lambda$, $c^\lambda$ and $c_0^\lambda$ as the sets of all $\lambda$-bounded, $\lambda$-convergent and $\lambda$-null sequences, respectively, that is

$$\ell^\lambda = \left\{ x \in w : \sup_n |\Lambda_n (x)| < \infty \right\},$$

$$c^\lambda = \left\{ x \in w : \lim_{n \rightarrow \infty} \Lambda_n (x) \text{ exists} \right\},$$

$$c_0^\lambda = \left\{ x \in w : \lim_{n \rightarrow \infty} \Lambda_n (x) \text{ exists} \right\},$$
\[
c_0^2 = \left\{ x \in w : \lim_{n \to \infty} \Lambda_n(x) = 0 \right\}
\]

where \( \Lambda_n(x) = \frac{1}{N_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) x_k \) \((k \in N)\).

H. Ganie and N. A. Sheikh [2] was introduced the spaces \( c_0(\Delta^2) \) and \( c(\Delta^2) \) as follows:

\[
c(\Delta^2) = \left\{ x \in w : \lim_{n} \Lambda_n(x) \text{ exists} \right\},
\]

\[
c_0(\Delta^2) = \left\{ x \in w : \lim_{n} \Lambda_n(x) = 0 \right\}
\]

where \( u = (u_k) \) is a sequence of complex numbers such that \( u_k \neq 0 \) for all \( k \in N \) and

\[
\Lambda_n(x) = \frac{1}{N_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) u_k (x_k - x_{k-1}) \quad (k \in N).
\]

2. Main results

Let \( u = (u_k) \) be a sequence of complex numbers such that \( u_k \neq 0 \) for all \( k \in N \). Then we introduce the sequence space

\[
\ell_\infty(\Delta^2) = \left\{ x = (x_k) \in w : \sup_{n} |\Lambda_n(x)| < \infty \right\}
\]

where \( \Lambda_n(x) = \frac{1}{N_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) u_k (x_k - x_{k-1}) \) \((k \in N)\). (2.1)

If \( u_k = 1 \) for all \( k \in N \), \( \ell_\infty(\Delta^2) \) reduces to \( \ell_\infty(\Delta^2) \) as follow:

\[
\ell_\infty(\Delta^2) = \left\{ x \in w : \sup_{n} |\Lambda_n(x)| < \infty \right\}
\]

where \( \Lambda_n(x) = \frac{1}{N_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) (x_k - x_{k-1}) \).

Furthermore we define the sequence space as follow:

\[
\ell_\infty^{(u)} = \left\{ x \in w : \sup_{n} |\Lambda_n(x)| < \infty \right\}
\]

where \( \Lambda_n(x) = \frac{1}{N_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) u_k x_k \).

Here and in sequel, we shall use the convention that any term with a negative subscript is equal to naught. e.g. \( \lambda_{-1} = 0 \) and \( x_{-1} = 0 \).

Let \( \lambda = (\lambda_k)_{k=0}^{\infty} \) be a strictly increasing sequence of positive reals tending to infinity, that is

\[
0 < \lambda_0 < \lambda_1 < \ldots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty.
\]
We define:

\[
\hat{\lambda}_{nk} = \begin{cases} 
\frac{(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)}{\lambda_n} u_k, & \text{if } k < n, \\
(\frac{\lambda_n - \lambda_{n+1}}{\lambda_n}) u_n, & \text{if } k = n, \\
0, & \text{if } k > n.
\end{cases}
\]

It is clear that the matrix \( \hat{\Lambda} = (\hat{\lambda}_{nk}) \) is a triangle. We shall assume that the sequences \( x = (x_k) \) and \( y = (y_k) \) are connected by the relation, that is \( y \) is \( \hat{\Lambda} \)-transform of \( x \), where

\[
y_k(\lambda) = \sum_{j=0}^{k-1} \left( \frac{\lambda_j - \lambda_{j+1}}{\lambda_k} \right) u_j x_j + \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} u_k x_k, \quad \text{for } k \in \mathbb{N}.
\]

Here and in what follows, the summation running from 0 to \( k - 1 \) is equal to zero when \( k = 0 \). It is clear from (2.1) that the relation (2.2) can be written as follows:

\[
y_k(\lambda) = \sum_{j=0}^{k-1} \left( \frac{\lambda_j - \lambda_{j+1}}{\lambda_k} \right) u_j (x_j - x_{j-1})
\]

for \( k \in \mathbb{N} \).

**Theorem 2.1:** The space \( \ell_\infty(\Delta^x) \) is a \( BK \)-space with the norm

\[
\|x\|_{\ell_\infty(\Delta^x)} = \|\hat{\Lambda}_n(x)\|_{\infty} = \sup_n |\hat{\Lambda}_n(x)|.
\]

**Proof:** The proof is seen easily, so we omitted.

One can easily check that the absolute property does not hold on the space \( \ell_\infty(\Delta^x) \) that is \( \|x\|_{\ell_\infty(\Delta^x)} \neq \|x\|_{\ell_\infty(\Delta^x)} \) where \( x = (|x_k|) \). Thus, the space \( \|x\|_{\ell_\infty(\Delta^x)} \) is \( BK \)-space of non-absolute type.

**Theorem 2.2:** The sequence space \( \ell_\infty(\Delta^x) \) is linearly isomorphic to the space \( \ell_\infty \), that is \( \ell_\infty(\Delta^x) \equiv \ell_\infty \)

**Proof:** To prove the theorem we must show the existence of linear bijection operator between \( \ell_\infty(\Delta^x) \) and \( \ell_\infty \). Hence, let define the linear operator with the notation (2.2), from \( \ell_\infty(\Delta^x) \) to \( \ell_\infty \) by \( x \to y(\lambda) = Tx \). Then \( T = y(\lambda) = \hat{\Lambda}(x) \in \ell_\infty \) for every \( x \in \ell_\infty(\Delta^x) \). Also, the linearity of \( T \) is clear. Further, it is trivial that \( x = 0 \) when ever \( Tx = 0 \) and hence \( T \) is injective.

Let \( y = (y_k) \in \ell_\infty \) and define the sequence \( x = (x(\lambda)) \) by

\[
x_k(\lambda) = \sum_{j=0}^{k} \sum_{i=j-1}^{\lambda} (-1)^{j-i} \frac{\lambda_i}{u_j(\lambda_j - \lambda_{j-1})} y_j \text{ for } k \in \mathbb{N}.
\]

Then, we obtain that

\[
x_k(\lambda) - x_{k-1}(\lambda) = \sum_{i=k-1}^{\lambda} (-1)^{i-k} \frac{\lambda_i}{u_k(\lambda_k - \lambda_{k-1})} y_i.
\]
Thus, for every $k \in \mathbb{N}$, we have by (2.1) that
\[
\hat{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (-1)^{k-i} \lambda y_j = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k y_k - \lambda_{k-1} y_{k-1}) = y_n
\]
This shows that $\hat{\Lambda}(x) = y$ and since $y \in \ell_\infty$, we obtain that $\hat{\Lambda}(x) \in \ell_\infty$. Thus we deduce that $x \in \ell_\infty (\Delta_\lambda^k)$ and $Tx = y$. Hence $T$ is surjective.

Further, we have for every $x \in \ell_\infty (\Delta_\lambda^k)$ that
\[
\|T(x)\|_\infty = \|\hat{\Lambda}(x)\|_\infty = \|x\|_{\ell_\infty (\Delta_\lambda^k)}
\]
Which means that $\ell_\infty (\Delta_\lambda^k)$ and $\ell_\infty$ are linearly isomorphic.

**Theorem 2.3:** The space $\ell_\infty (\Delta_\lambda^k)$ is a $BK$-space with the norm
\[
\|x\|_{\ell_\infty (\Delta_\lambda^k)} = \sup_n |\hat{\Lambda}_n(x)|.
\]
**Proof:** The proof is seen easily, so we omitted.

**3. The inclusion relations**

**Theorem 3.1:** The inclusion $c(\Delta_\lambda^k) \subset \ell_\infty (\Delta_\lambda^k)$ strictly hold.

**Proof:** Let $x \in c(\Delta_\lambda^k)$. Then $\hat{\Lambda}(x) \in c$. Further more, since $c \subset \ell_\infty$ we have $\hat{\Lambda}(x) \in \ell_\infty$. Hence $x \in \ell_\infty (\Delta_\lambda^k)$.

It is clear that the inclusion $c(\Delta_\lambda^k) \subset \ell_\infty (\Delta_\lambda^k)$ hold. To show that the inclusion is strict, consider the sequence $x = (x_k)$ and $x_k - x_{k-1} = (-1)^k (\lambda_k + \lambda_{k-1}) - (\lambda_k - \lambda_{k-1})u_k$ for all $k \in \mathbb{N}$. Thus we have that
\[
\hat{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (-1)^k (\lambda_k + \lambda_{k-1}) - (\lambda_k - \lambda_{k-1})u_k
\]
for every $n \in \mathbb{N}$.

This shows that $\hat{\Lambda}_n(x) \in \ell_\infty$, but $\hat{\Lambda}_n(x) \notin c$. Thus, the sequence $x$ is in $\ell_\infty (\Delta_\lambda^k)$, but not in $c(\Delta_\lambda^k)$.

Hence, the inclusion $c(\Delta_\lambda^k) \subset \ell_\infty (\Delta_\lambda^k)$ strictly holds. This completes the proof.

**Theorem 3.2:** The inclusion $\ell_\infty^c \subset \ell_\infty (\Delta_\lambda^k)$ holds.

**Proof:** Let $x \in \ell_\infty^c$. Then we deduce that
\[
\left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \Delta x_k \right| \leq \frac{1}{\lambda_n} \sum_{k=0}^{n} \left| (\lambda_k - \lambda_{k-1}) \Delta x_k \right| + \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_{k-1}.
\]
Hence $x \in \ell_\infty (\Delta_\lambda^k)$.

**Theorem 3.3:** The inclusion $\ell_\infty (\Delta) \subset \ell_\infty (\Delta_\lambda^k)$ holds.

**Proof:** Let $x \in \ell_\infty (\Delta)$. Hence we deduce that
\[
\left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \Delta x_k \right| \leq \frac{1}{\lambda_n} \sup_k |\Delta x_k| \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) = \sup_k |\Delta x_k|.
\]
Therefore, we derive $x \in \ell_\infty(\Delta^\lambda)$.

**Theorem 3.4:** The inclusion $\ell_\infty \subseteq \ell_\infty(\Delta^\lambda)$ strictly holds.

**Proof:** The inclusion $\ell_\infty \subseteq \ell_\infty(\Delta^\lambda)$ holds [see 5]. We derived from Theorem 3.2 that $\ell_\infty^\lambda \subset \ell_\infty(\Delta^\lambda)$. Hence $\ell_\infty \subseteq \ell_\infty(\Delta^\lambda)$. Further, consider the sequence $y = (y_k)$ defined by

$$y_k = \sqrt{k+1}; \quad (k \in N).$$

Then, it is trivial that $y \notin \ell_\infty$. On the other hand, we have that $\lambda \in \ell_\infty(\Delta^\lambda)$ from in equality;

$$\sup_n \left| \frac{1}{n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \Delta y_k \right| \leq \sup_n \left| \Delta y_k \right| \sup_n \left| \frac{1}{n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \right| \leq 1$$

Thus, the sequence $y$ is in $\ell_\infty(\Delta^\lambda)$ but not in $\ell_\infty$. We therefore deduce that the inclusion $\ell_\infty \subseteq \ell_\infty(\Delta^\lambda)$ is strict. This concludes the proof.

**Theorem 3.5:**

i. For $|u_k| \leq 1$ for all $k \in N$, the inclusion $\ell_\infty(\Delta^\lambda) \subseteq \ell_\infty^\lambda(\Delta^\lambda)$ holds.

ii. For $|u_k| > 1$ for all $k \in N$, the inclusion $\ell_\infty(\Delta^\lambda) \subseteq \ell_\infty(\Delta^\lambda)$ holds.

**Proof:** (i) Let $x \in \ell_\infty(\Delta^\lambda)$. Then

$$\left| \frac{1}{n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \Delta x_k \right| < \infty.$$ Since

$$\left| \frac{1}{n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) u_k \Delta x_k \right| \leq \frac{1}{n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |u_k| \left| \Delta x_k \right| \leq \frac{1}{n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \left| \Delta x_k \right| < \infty,$$

We have $\ell_\infty(\Delta^\lambda) \subseteq \ell_\infty(\Delta^\lambda)$.

(ii) Let $x \in \ell_\infty(\Delta^\lambda)$. From in equality

$$\left| \frac{1}{n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \Delta x_k \right| \leq \frac{1}{n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |u_k| \left| \Delta x_k \right| < \infty.$$ We have that $\ell_\infty(\Delta^\lambda) \subseteq \ell_\infty(\Delta^\lambda)$.

**Theorem 3.6:**

i. For $|u_k| \leq 1$ for all $k \in N$, the inclusion $\ell_\infty^\lambda \subseteq \ell_\infty^k(\Delta^\lambda)$ holds.

ii. For $|u_k| > 1$ for all $k \in N$, the inclusion $\ell_\infty^k(\Delta^\lambda) \subseteq \ell_\infty^\lambda(\Delta^\lambda)$ holds.
Proof: (i) Let \( x \in \ell_\infty^4 \). Then

\[
\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |u_k| |x_k| \leq \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |x_k| < \infty.
\]

We have \( \ell_\infty^4 \subseteq \ell_\infty^{\lambda,n} \).

(ii) Let \( x \in \ell_\infty^{\lambda,n} \). Then we derive that

\[
\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |x_k| \leq \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |u_k| |x_k| < \infty.
\]

Hence \( \ell_\infty^{\lambda,n} \subseteq \ell_\infty^4 \).

References


