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Abstract

In this paper we carried out a computational study of MHD flow and heat transfer analysis in a cylindrical pipe field with porous material, the momentum equation were solved using the Galerkin weighted residual method while the energy equation via the semi implicit finite difference method. Results obtained showed that an increase in the Darcy number shows an increase in the velocity profile while an increase in the Brinkman number leads to increase in the temperature profile.

Keywords magnetohydrodynamics, weighted residual method, porous media

Introduction

MHD problems arise in a wide variety of situations ranging from the explanation of the origin of the earth magnetic field and the prediction of space weather to the damping of turbulent fluctuations in semiconductor melts during crystal growth and even the measurement of the flow rates of beverages in the food industry, which leads to research interest by several authors such the work of (Smith (1971), Branover and Gershon (1976), Holroyd (1979), and Holroyd (1980)). Hartmann (1937) studied the influence of a transverse uniform magnetic field on the flow of viscous incompressible electrically conducting fluid between two finite parallel stationary and insulating plates.

The viscous dissipation heat in the natural convective flow is important, when the flow field is of extreme size or at low temperature or in high gravitational field. Examples of flow situations where the dissipation is important includes high shear flows (such as thin lubricant films), in boundary layers in high speed supersonic flight and in large volume channel flows.

Several works has been published on the effect of viscous dissipation, these includes the work ofOlaseeni and Oladapo (2010), examined the effect of viscous dissipation on the temperature profile of a laminar flow in a channel filled with saturated porous media. Ahmed and Batin(2010) who studied the analytical model of MHD mixed convective radiating fluid with viscous dissipative heat, Rao and Babu(2010) who carried out finite element analysis of radiation and mass transfer flow past semi-infinite moving vertical plate with viscous dissipation.Okedayo et al.(2011), investigated the effects of viscous dissipation, constant wall temperature and a periodic pressure field on unsteady flow through a horizontal channel filled with a porous material. The
coupled non-linear differential equations governing the flow were solved analytically using the usual method of separation of variables and simple perturbation techniques.

The porous media heat transfer problems have numerous thermal engineering applications such as geothermal energy recovery, crude oil extraction, thermal insulation, ground water pollution, oil extraction, thermal energy storage, thermal insulations, and flow through filtering devices. Hamad and Bashir(2011).

The method of weighted residuals is a tool for obtaining simple and accurate approximation techniques for solving differential equations. The method of weighted residual (MWR) actually encompasses several methods including collocation, Galerkin, and the least squares. Several works on the weighted residual methods exist in literature such as the work of Hatami et al(2014) who carried out a Computer simulation of MHD blood conveying goldnanoparticles as a third grade non-Newtonian nanofluid in a hollow porous vessel using the Galerkin’s weighted residual method and Moakher et al(2015) who studied the MHD fluid flow of fourth grade fluid through the channel with slip condition using the collocation weighted residual method.

In this research paper we carried out a numerical study of flow and heat transfer analysis in a pipe filled with porous media using a combined Galerkin weighted residual and the semi implicit finite difference method, the weighted residual method was applied to the momentum equation while the finite method to the energy equation.

**Problem formulation**

Consider an unsteady flow of a Newtonian fluid in a cylindrical pipe filled with a porous material with an axial magnetic field with velocity \( u(r, t) \) where \( r = \text{radius of the pipe and } t = \text{time} \) the momentum and energy equations are given by

\[
\begin{align*}
\rho \frac{\partial u}{\partial t} &= \mu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) - \frac{\mu}{k} u - \sigma B_0^2 u \\
\rho C_p \frac{\partial T}{\partial t} &= \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \mu \left( \frac{\partial u}{\partial r} \right)^2 \\
\frac{\partial u(0, t)}{\partial r} &= 0, \quad \frac{\partial T(0, t)}{\partial r} = 0, \quad T(R, t) = T_0, \quad u(r, 0) = 0, \quad T(r, 0) = T_0
\end{align*}
\]

Introducing the following non-dimensional variables we have

\[
U = \frac{u}{U_s}, \quad \tau = \frac{t U_s}{R}, \quad \theta = \frac{T - T_0}{T_0}, \quad r = \frac{r}{R}
\]
\[ \frac{\partial U}{\partial \tau} = \frac{1}{\text{Re}} \left( \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} \right) - \frac{1}{\text{Da}} U - \text{Ha}^2 U \]
\[ \frac{\partial \theta}{\partial t} = \frac{1}{\text{Pr}} \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) + \text{Br} \left( \frac{\partial U}{\partial r} \right)^2 \]
\[ U(1, \tau) = 1, \quad \frac{\partial U(0, \tau)}{\partial r} = 0, \quad \frac{\partial \theta(0, \tau)}{\partial r} = 0, \quad \theta(1, \tau) = 0, \quad U(r, 0) = 0, \quad \theta(r, 0) = 0 \]  

where,
\[ \text{Re} = \frac{\rho U L}{\mu}, \quad \text{Da} = \frac{\rho U_s k}{\rho \mu}, \quad \text{Ha}^2 = \frac{R \sigma B_0^2}{\rho U_s}, \quad \text{Pr} = \frac{\rho c_p R U_s}{\alpha}, \quad \text{Br} = \frac{\mu U_s}{\rho c_p T_0 R} \]

Are the Reynolds number, Darcy number, Hartmann number, Prandtl number and the Brinkman number respectively.

**Problem solution**
We proceed to solve the momentum equation by the Galerkin weighted residual method with the following algorithm.

Consider a differential operator \( L \) acting on a function \( u \) to produce a function \( f \).
\[ L[u(r)] = f(r) \]  

The objective is to approximate \( u(r) \) by a function \( \bar{u}(r) \), which is a linear combination of basis functions chosen from a linearly independent set of the form
\[ \bar{u}(r, \tau) = \sum_{i=0}^{n} c_n(\tau) \phi_n(r) \]

Substituting this into the momentum equation we obtain the residual equation of the type
\[ R(c_i, r) = L \left[ \sum_{i=0}^{n} c_n(\tau) \phi_n(r) \right] - f(r) \]

It is required that the residuals be orthogonal to a set of weight functions and in this case to the chosen basis functions \( \phi_n(r) \). Which results into a system of first order ordinary differential equations of the form
\[ c_n'(\tau) = A_n g(\tau) + B_n \]

Which are then solved for the \( c_n(\tau) \)'s, hence we select basis function of the form
\[ u(r, \tau) = 1 + a_0(\tau)(r^2 - 1) + a_1(\tau)(r^3 - 1) \]

Which satisfies the given set of boundary conditions.

By substituting the solution of the momentum equation into the energy equation we have equation of the form
\[ \frac{\partial \theta}{\partial t} = \frac{1}{\text{Pr}} \left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) + F(r, \tau) \]
Which is an inhomogeneous partial differential equation. Which shall be solved the semi implicit finite difference scheme as follows:

\[
\frac{\partial \theta}{\partial \tau} = \frac{\theta_{j+1}^n - \theta_j^n}{\Delta \tau}, \quad \frac{\partial \theta}{\partial r} = \frac{\theta_j^n - \theta_{j-1}^n}{2\Delta r}, \quad \frac{\partial^2 \theta}{\partial r^2} = \frac{\theta_{j+1}^n - 2\theta_j^n + \theta_{j-1}^n}{\Delta r^2}
\]

Which results into the finite difference scheme

\[
(r_j Pr - 2\lambda r_j)\theta_j^{i+1} - r_j Pr \theta_j^i = \lambda r_j \theta_{j+1}^i - \lambda r_j \theta_j^{i-1} + \lambda r_j \theta_{j+1}^i + \lambda r_j \theta_j^{i-1} + \Delta r_j Br Pr F_j^i
\]

where

\[
\lambda = \frac{\Delta \tau}{\Delta r^2}, \quad \lambda_i = \frac{\Delta \tau}{2\Delta r}
\]

We divide the interval \(0 \leq r \leq 1\) into \(n\) equal sub-intervals with \(n-1\) internal mesh points and \(n-1\) system of equations to be solved. The system of equation can be written as a tri-diagonal matrix which is inverted to obtain the values \(\theta_j^i\)'s at the mesh points.

**Results and Discussion**

The result of the Galerkin weighted residual method is given by

\[
a_0(t) = \frac{-348.6434159G \text{Re} \exp\left[\frac{-5.87154117 + G \text{Re} \text{Re}}{763.300352 + 130G \text{Re}}\right] \tau}{798.6434162G \text{Re} \exp\left[\frac{-29.71307421 + G \text{Re} \text{Re}}{3862.699648 + 130G \text{Re}}\right] \tau} + \frac{-225G^2 \text{Re}^2 + 2835G \text{Re}}{2313G \text{Re} + 65G^2 \text{Re}^2 + 11340}
\]

\[
a_1(t) = \frac{1.109226967G \text{Re}(-40897.27524G^2 \text{Re}^2 - 1.455313810 \times 10^6G \text{Re} - 7.135001557 \times 10^6) \exp\left[\frac{-29.71307421 + G \text{Re} \text{Re}}{2313G \text{Re} + 65G^2 \text{Re}^2 + 11340}(3862.699648 + 130G \text{Re})\right] \tau}{(2313G \text{Re} + 65G^2 \text{Re}^2 + 11340)(3862.699648 + 130G \text{Re})} + \frac{0.482269668G \text{Re}(-18512.72476G^2 \text{Re}^2 - 6.587681904 \times 10^5G \text{Re} - 3.229758443 \times 10^5) \exp\left[-(5.87154117 + G \text{Re}) \right] \tau}{(2313G \text{Re} + 65G^2 \text{Re}^2 + 11340)(763.300352 + 130G \text{Re})} + \frac{280G^2 \text{Re}^2}{2313G \text{Re} + 65G^2 \text{Re}^2 + 11340}
\]

In order to investigate the effect of the various thermo-physical parameters arising in the flow we take \(\Delta \tau = 0.1\) and \(\Delta r = 0.1\).

In figure.1 we plot the velocity profile for various values of the Darcy number(Da) for \(t=1, \text{Re} = 10, \text{H} = 1\), it is observed that increase Darcy parameter leads to increase in velocity profile, that is the porosity increases which enhances the fluid flow, at Darcy 0.1 we experience a retarded flow after which velocity increases. While in figure.2 we show the velocity profile for different values of the Hartman number when \(t=1, \text{Re} = 1, \text{Da} = 0.1\), velocity decreases with increasing Hartman number due to the strength of the Lorentz force. In figure .3 we depicts the velocity profile for variation of Reynold’s number when \(t=1, \text{H} = 1, \text{Da} = 0.1\) the flow experienced a gradual decrease with a retarded flow and afterwards increases.
Figure.4 shows the temperature profile for various values of the Prandtl number. It is observed that temperature increases with increasing Prandtl number. Figure.5 shows the temperature profile for various values of Brinkman number and it is observed that the temperature profile increases with increase in the value of the Brinkman number which shows that heat was added to the system.

Figure.1: Velocity Profiles for Various Values of the Darcy.

Figure.2: Velocity Profiles for Various Values of the Hartman Number.
Figure 3: Velocity Profiles for Various Values of the Reynold’s number ($Re$).

Figure 4: Temperature profile for various values of the Prandtl Number.
6.1 Conclusion
This study presents a numerical study of the flow of an incompressible fluid through a horizontal channel filled with porous media with viscous dissipation effect. The governing equations were solved numerically using Galerkin weighted residual method and semi-implicit finite difference scheme. The result obtained were analyzed for various thermo-physical parameter entering into the dimensionless equations.

The following were observed:
1. Increase in Darcy number leads to increase in velocity profile.
2. Increase in Hartman number leads to decrease in velocity.
3. Increase in Reynold’s number leads increase of the velocity.
4. Increase in Prandtl number leads to increase in the temperature profile.
5. Increase in Brinkman number result to increase in temperature profile which shows that heat was added to the system.

References


