ON THE STEADY MHD POISEUILLE FLUID FLOW BETWEEN TWO INFINITE PARALLEL POROUS PLATES

D.W. Kiema¹*, W.A. Manyonge ², J.K. Bitok ³.

¹ Department of Mathematics and Physical Sciences
Pwani University,
P.O. Box 195 - 80108 Kilifi, KENYA

² Centre for Research for New and Renewable Energies
Maseno University,
P.O. Box 333, Maseno, KENYA

³ Department of Mathematics and Computer Science
University of Eldoret,
P.O. Box 1125, Eldoret, KENYA.

*corresponding author

ABSTRACT

In this paper, we examine the motion of two dimensional steady laminar flow of a viscous Magnetohydrodynamic incompressible fluid between two infinite parallel porous plates under the influence of uniform transverse magnetic field and with constant pressure gradient. Both the lower plate and the upper plates are assumed porous and the fluid enters the flow region through the lower plate and leaves through the upper plate with constant velocity \( v_u \). The resulting coupled differential equations are solved by using finite difference approach. The resulting block tri-diagonal system is solved using Thomas-algorithm and the velocity profiles obtained expressed in terms of Hartmann number.

Key words: MHD flow, Poiseuille flow, numerical methods, Hartmann number, uniform transverse magnetic field.

1. Introduction

Magnetohydrodynamics (MHD) is the fluid mechanics of electrically conducting fluids. The theoretical study of flows in porous media and MHD fluid flows has been on recent years of great interest due to its several applications in geothermal, oil reservoir engineering, separation of matter from fluids, MHD power generation, aerodynamics, astrophysics and environmental applications. Internal flows of MHD fluid in ducts and channels filled with porous media have received special attention. Such problems are called transpiration cooling and are effective in reducing heat transfer between fluid and boundary layer with much application to cooling of rockets and jets. When an electrically conducting fluid flows through a magnetic field the interaction between the electromagnetic field and hydrodynamics produces magnetohydrodynamics. Some of these fluids include liquid metals such as mercury and molten iron while ironized gases also known as plasma, an example being the Solar atmosphere. MHD in its present form is due to pioneer work of some authors such as Swedish electrical engineer Hannes Alfvén [1] in 1942, Shercliff [2], Cowling [3]
and Sinha [6]. Palm et al. [7] investigated on the steady free convection in a porous medium and also extended their research work into heat dispersion effect on steady convection in an isotropic porous media. Raptis et al. [9] studied hydromagnetic free convective flow through a porous medium between two parallel plates. Chandra and Prasad [10] analysed the pulsatile flow in circular tubes of varying cross section with permeable wall. Suction and ejection are permissible for the fluid velocity through the wall and flow in a direction normal to the wall. Singh [11] discussed unsteady flow of liquid through a channel with pressure gradient changing exponentially under the influence of inclined magnetic field and solved this by the method of laplace Transform. Al-Hadhrami [12] analysed flow of fluids through horizontal channels of porous materials and obtained velocity expressions in terms of Reynolds number while Ganesh et al.[13 ] considered unsteady MHD Stokes flow of viscous fluid between two parallel porous plates. They analysed fluid being withdrawn through both walls of the channel at the same rate. Manyonge et al. [14] studied two dimensional Poiseuille flow of an electrically conducting fluid between parallel plates under the influence of transverse magnetic field under a constant pressure gradient and assessed the effect to velocity if the lower plate was porous while the upper plate was not. The resulting differential equation was solved by analytical method and the solution expressed in terms of Hartmann number. In this paper, we examine laminar MHD steady incompressible fluid between two infinite parallel porous plates under the influence of uniform transverse magnetic field. Both the lower plate and the upper plates are assumed porous as the fluid enters the flow region through the lower plate and leaves through the upper plate with constant velocity $v_o$. The resulting coupled differential equations are solved by using finite difference approach. The resulting block tri-diagonal system is solved using Thomas’-algorithm and the velocity profiles obtained expressed in terms of Hartmann number for various angles of inclination.

2. Mathematical Formulation

The basic concept describing magnetohydrodynamics phenomena can be described by considering an electrically conducting fluid moving with a velocity vector $\mathbf{V}$. At right angle to this, we apply a magnetic field, $\mathbf{B}_{app}$. We then assume that steady flow conditions have been attained i.e. flow variables are independent of the time $t$. This condition is purely for analytic reasons so that no macroscopic charge density is being built up at any place in the system as well as all currents are constant in time. Because of the interaction of two fields, namely, velocity and magnetic fields, an electric field denoted by $\mathbf{E}$ is induced at right angles to both $\mathbf{V}$ and $\mathbf{B}_{app}$ (see figure 1 below). This electric field is given by

$$E_{ind} = V \times B_{app}$$

(1)
We assume that the conducting fluid is isotropic in spite of the magnetic field and denote its electrical conductivity by the scalar quantity $\sigma$. By Ohms law, the density of the current induced in the conducting fluid is denoted by $J_{\text{ind}}$ and is given by

$$J_{\text{ind}} = \sigma E_{\text{ind}}$$

or we can simply write this as

$$J_{\text{ind}} = \sigma (V \times B_{\text{app}})$$

Simultaneously occurring with the induced current is the induced ponderomotive force or the Lorentz force $F_{\text{ind}}$ which is given by

$$F_{\text{ind}} = J_{\text{ind}} \times B_{\text{app}}$$

The Lorentz force is significant in determining the flow profile based on the dimensionless Hartmann number which is given by the ratio of the magnetic body force and the viscous force i.e. $Ha = (N\cdot Re)^{1/2}$, where $N = Ha^2/Re = \sigma \cdot L \cdot B^2 / \rho \cdot U$ stands for the nondimensional interaction parameter known as Stuart number which is defined as the ratio of electromagnetic to inertial forces, and this gives an estimate of the relative importance of a magnetic field of the flow. It is also relevant for flows of conducting fields e.g. in fusion reactors, steel casters or plasmas. On the other hand, $Re = UL / \nu$ is the nondimensional hydrodynamic Reynolds number, so the Hartmann number can be rewritten as $Ha = L \cdot B \cdot (\sigma / \mu)^{1/2}$ where $\mu$ is the dynamic viscosity and $\nu$ is kinematic viscosity.

The Lorentz force will occur because, as an electric generator, the conducting fluid cuts the lines of the magnetic field. The vector $F$ is the vector cross product of both $J$ and $B_{\text{app}}$ and is a vector perpendicular to the plane of both $J$ and $B_{\text{app}}$. The induced force is parallel to $V$ but in opposite direction. Laminar flow through a channel under uniform transverse magnetic field is important because of the use of MHD generator, MHD pump, crude oil purification and electromagnetic flow meter.

We now consider an electrically conducting viscous, steady, incompressible fluid moving between two infinite parallel plates both of which are kept at a constant distance $2h$ between them. The upper plate and the lower plate are kept stationary. The fluid is acted upon by a constant pressure gradient which makes this flow a plane poiseuille flow.

For zero displacement and Hall currents, Maxwell’s equations together with Ohms law and Law of magnetic conservation are written as :

$$\nabla \times E = -\frac{\partial B_{\text{app}}}{\partial t} \quad \text{or} \quad \nabla \times E + \frac{\partial B_{\text{app}}}{\partial t} = 0$$

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

$$J = \sigma (E + V \times B_{\text{app}})$$

$$\nabla \cdot B_{\text{app}} = 0$$

$$\nabla \cdot D = 0$$

where $H$ is the magnetic field intensity vector and $D$ is the electric displacement vector which is analogous to magnetic vector $B_{\text{app}}$.

The governing equations for the flow of incompressible Newtonian fluid that we use in this study are the continuity equation

$$\nabla \cdot V = 0$$

and the momentum equation :-

$$\rho \left[ \frac{\partial V}{\partial t} + (V \cdot \nabla) V \right] = -\nabla p + \mu \nabla^2 u + J \times B_{\text{app}}$$
where $\rho$ is the fluid density, $p$ is the fluid pressure function, and $J \times B_{agy}$ is the Lorentz force. Navier-Stokes equations are differential equations that determine the velocity of the fluid at any instant of time, while Maxwell’s equations are differential equations that combine together to form complex equations either magnetic or electric field or both.

In the present analysis, the following important assumptions are made:

i. The fluid flow is incompressible.

ii. The fluid flow is steady hence the flow variables do not depend on time.

iii. The fluid is electrically neutral i.e. there is no surplus electrical charge distribution present in the fluid.

iv. The only body forces present are Lorentz forces.

v. The fluid flow is unidirectional in $x$-axis, the channel formed by the two plates.

vi. The flow is laminar i.e. the flow paths of individual particles of the fluid do not cross those of neighbouring particles, hence, making it possible to follow the path/motion of every individual particle.

The equation of continuity and the momentum equations in two dimensions together with the above assumptions reduces (9) and (10) to the form:

\[
\frac{\partial v}{\partial y} = 0 \quad (11)
\]

\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} + \frac{F_x}{\rho} \quad (12)
\]

\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (13)
\]

where $F_x$ is the component of the magnetic force in $x$-direction. From (11) this implies $v = \text{constant}$ or $v = 0$, and since from the flow geometry $v$ cannot be a constant therefore, we choose $v = 0$. From (13) we find that pressure does not depend on $y$. Hence $p$ is a function of $x$ alone.

3. Non-dimensionalizing of the governing equations

Equations (12) can be non-dimensionalized using the characteristic velocity $U$, the body length $L$ by denoting the dimensional quantities given as

\[
x = \frac{x}{\bar{L}}, \quad y = \frac{y}{\bar{L}}, \quad p = \frac{pL^2}{\rho \nu^2}, \quad u = \frac{uL}{\nu} \quad (14)
\]

and subsequently solving subject to boundary condition $\bar{u} = 0$ when $\bar{y} = \pm L$, where bars denote dimensionless quantities.

Using assumptions (3), (4) and (5), we note that $B_x = B_z = 0$ and $\bar{v} = \bar{w} = 0$ so that, $V_x = \bar{u}$ and $B_{agy} = B_z j$ where $B_z$ is the magnetic field strength component assumed to be applied to a direction perpendicular to fluid motion in $\bar{y}$-direction, $i$ and $j$ are unit vectors in the $\bar{x}$ and $\bar{y}$-directions respectively. Now, $F_x = \sigma \left[ (\bar{u} \times B_z j) \right] \times B_z j$ from which we find that,

\[
\frac{F_x}{\rho} = -\frac{\sigma}{\rho} B_z^2 \bar{u}. \quad (15)
\]
Using the above dimensionless quantities, equation (12) reduces to
\[ 0 = \frac{1}{\rho} \frac{\partial \tilde{p}}{\partial \tilde{x}} + \nu \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} - \frac{\sigma}{\rho} B_o^2 \tilde{u} \]
or
\[ \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} - \frac{\sigma}{\mu} B_o^2 \tilde{u} = \frac{\partial \tilde{p}}{\partial \tilde{x}} \]

Or
\[ \frac{d^2 \tilde{u}}{d \tilde{y}^2} - \frac{\sigma}{\mu} B_o^2 \sin(\alpha) \tilde{u} = \frac{1}{\mu} \frac{d \tilde{p}}{d \tilde{x}} \]  \hspace{1cm} (16)

Where \( \alpha \) is the angle between \( V \) and \( B_{app} \) which means that, the two fields can be assessed at any angle \( \alpha \) for \( 0 \leq \alpha \leq \pi \).

Differentiating equation (16) w.r.t. \( x \) we obtain
\[ \frac{d^2 \tilde{p}}{d \tilde{x}^2} = 0 \]
and on integration this we obtain
\[ \frac{d \tilde{p}}{d \tilde{x}} = -C \] \hspace{1cm} (a constant). With this in mind and dropping the bars \( \mu \) (for convinence) we get
\[ \frac{d^2 u}{d y^2} - \frac{\sigma}{\mu} B_o^2 \sin(\alpha) u = \frac{1}{\mu} \frac{d \tilde{p}}{d \tilde{x}} \]  \hspace{1cm} or
\[ \frac{d^2 u}{d y^2} = M^2 u - \frac{1}{\mu} \frac{d \tilde{p}}{d \tilde{x}} = 0 \]  \hspace{1cm} (17)

Where \( M = M^* \sin \alpha \) and \( M^* = \frac{\sigma}{\mu} B_o \) \( \frac{L^2}{h} \) = \( Ha \) and \( Ha \) is the Hartmann number given by
\[ Ha^2 = \frac{\sigma B_o^2 \mu}{\mu} \]
Hence equation (17) can be rewritten as
\[ \frac{d^2 u}{d y^2} - M^2 u + C = 0 \]  \hspace{1cm} (18)

Whose solution subject to boundary conditions
\[ u = 0 \quad y = -1 \]
\[ u = 0 \quad y = +1 \] \hspace{1cm} (BC)
was given by Singh (1992) by method of solution of differential equation with constant coefficients as
\[ \frac{u}{C} = \frac{1}{M^2} \left[ 1 - \frac{\cosh M y}{\cosh M} \right] \]  \hspace{1cm} (19)

4. MHD Fluid Flow between Two Infinite Parallel Porous Plates.

We now consider the MHD steady laminar flow of viscous incompressible fluid between two infinite parallel porous plates separated by a distance \( 2h \) and \( x \)-axis be taken in the middle of the channel parallel to the direction of flow, the \( y \) direction perpendicular to the flow, and the width of the plates parallel to the \( z \)-direction. The word infinite here means that the width of the plates is large compared with \( h \) and hence we treat the flow to be two dimensional. We also take the velocity component \( w \) to be zero everywhere and \( u \) as function of \( y \) alone.

Since both plates have very fine holes distributed uniformly over the entire surface of the plates through which the fluid can flow freely and continously, the fluid will enter the flow region through the lower plate and leave through the upper plate with constant characteristic velocity \( v \), along \( y \)-
direction. For the present steady flow the equation of continuity reduces to \( \frac{\partial v}{\partial y} = 0 \), so that \( v \) does not vary with \( y \). Similarly, the \( x \) and \( y \) momentum equations are given by

\[
v_o \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \frac{\partial^2 u}{\partial y^2}
\]

(20)

\[0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad \text{(21)}\]

Equation (21) shows that pressure does not depend on \( y \) and therefore the equation collapses as \( p \) is a function of \( x \) alone and so equation (20) reduces to

\[
\frac{dp}{dx} = \rho \left[ -v \frac{\partial^2 u}{\partial y^2} - v_o \frac{du}{dy} \right]
\]

(22)

Differentiating equation (22) w.r.t. \( x \) we obtain \( \frac{d^2 p}{dx^2} = 0 \) or \( \frac{d}{dx} \left( \frac{dp}{dx} \right) = 0 \).

Integrating, \( \frac{dp}{dx} = -P \) (a constant-say), where the negative sign has been taken to show pressure decreases as \( x \) increases. Substituting this, equation (22) now becomes

\[
\frac{d^2 u}{dy^2} - v_o \frac{du}{dy} = -\frac{P}{\rho v}
\]

(23)

If the fluid is subjected to uniform transverse magnetic forces, we now model equation (23) by adding the term \( -M^2 u \) to yield

\[
\frac{d^2 u}{dy^2} - v_o \frac{du}{dy} + \frac{P}{\mu} - M^2 u = 0
\]

(24)

Let equation (24) be of the form \( \frac{d^2 u}{dy^2} - A \frac{du}{dy} - M^2 u + B = 0 \)

(25)

where \( A = \frac{v_o}{v} \) and \( B = \frac{P}{\mu} \) are constants of the fluid and solve equation (25) by finite difference method under the initial and the boundary conditions

\[
u = 0 \quad y = -1 \quad u = 0 \quad y = +1
\]

(26)

5. Numerical Solution of the Governing Equation

The non-linear differential equations (25) subject to initial and boundary conditions (26) is solved using finite difference approach. In this technique derivatives occurring in the generated differential equations have been replaced by their finite difference approximations. An iterative scheme is used to solve the linearized system of difference equations. Central difference approximations have been used because they are more accurate than forward and backward differences. The numerical computation of the generated linearized system of equations based on our step size and results of these are achieved with the aid of MATLAB application software.
Representing the step size by \( k \), the finite difference equation corresponding to equation (25) is given as

\[
\frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} - A \frac{U_{i,j+1} - U_{i,j-1}}{2k} - M^2 U_{i,j} + B = 0
\]  

(27)

where \( i \) and \( j \) stands for increments in \( t \) and \( y \) respectively. The block tri-diagonal system is solved using Thomas’ algorithm. All calculations have been carried out for \( A = 1 \), \( B = 2 \) and \( k = 0.25 \). Velocity profiles for Hartmann numbers \( M^* = 0.5 \), \( M^* = 1.5 \), \( M^* = 2.5 \) and angle of inclinations \( \alpha = 15^\circ \), \( \alpha = 30^\circ \) and \( \alpha = 45^\circ \) are presented.

6. Results and discussions
Numerical calculations have been performed for velocity profiles. The results are presented graphically in figures (2)-(4) for various Hartmann numbers and different angles of inclinations.

![Figure 2: Velocity profiles for various Hartmann numbers for angle of inclination 15°](image)
Figure 3: Velocity profiles for various Hartmann numbers for angle of inclination $30^0$

Figure 4: Velocity profiles for various Hartmann numbers for angle of inclination $45^0$
7. Conclusions

The steady MHD poiseuille fluid flowing between two infinite parallel porous plates under the influence of tranverse magnetic field and with constant pressure gradient has been investigated. The results from the figures (2)-(4) shows how velocity of the fluid changes with varied Hartmann numbers. An increase in the Hartmann number leads to a decrease in velocity distribution. This is due to Lorentz force generated by the application of constant inclined magnetic field which offers resistance opposing the fluid motion and hence decreasing the flow.

REFERENCES